Exact solutions for the field electron emission achieved from a flat metal using the standard Fowler-Nordheim equation with a correction factor that accounts for the electric field, the work function and the Fermi energy of the emitter

Mayer, Alexandre

Published in:
Journal of Vacuum Science & Technology B

DOI:
10.1116/1.3562965

Publication date:
2011

Document Version
Early version, also known as pre-print

Exact solutions for the field electron emission achieved from a flat metal using the standard Fowler-Nordheim equation with a correction factor that accounts for the electric field, the work function and the Fermi energy of the emitter

A. Mayer*

Laboratoire de Physique du Solide, University of Namur-FUNDP,

Rue de Bruxelles 61, B-5000 Namur, Belgium

We use a transfer-matrix technique to simulate field electron emission from a flat metal. We compare in particular the results provided by this numerical scheme with those predicted by the standard Fowler-Nordheim equation. This comparison aims at establishing the influence of different approximations introduced in the standard Fowler-Nordheim theory (in particular the use of the Jeffreys-Wentzel-Kramers-Brillouin approximation for evaluating the transmission coefficient of the surface barrier and the series-expansion of this coefficient when integrating over the normal-energy distribution of the incident electrons). In addition to the field and work function considered in previous work, we explore the dependence of the emission current on the Fermi energy of the emitter. This physical parameter, which is related to the density of free carriers in the emitter, does not appear in the final form of the standard Fowler-Nordheim equation. It is therefore discarded from most analysis of field-emission data. We show however by a series of arguments that the emission currents are affected by the Fermi energy of the emitter. We finally establish a correction factor to be used with the Murphy-Good expression that accounts for the field, for the work function and for the Fermi energy of the emitter and provides the exact solution for the emission achieved from a flat metal.

* Electronic address: alexandre.mayer@fundp.ac.be

I. INTRODUCTION

Field electron emission is nearly as old as quantum mechanics. It remains however a subject of actuality in fundamental science and for the development of technologies.[1] In the cold-emission regime in which the thermal excitation of electrons to energies that are above the surface barrier of the emitter can be neglected, this emission process is actually due to the quantum-mechanical tunneling of electrons through the surface barrier. The work function of the emitter essentially determines the energy at which this process is taking place, while the role of the external field is essentially to reduce both the height and the width of the surface barrier. The Fermi energy of the emitter, which is related to the density of free carriers, is usually not considered as a significant parameter.

The first successful modeling of field electron emission from a flat metal is attributed to Fowler and Nordheim.[2] Their analysis was restricted to a triangular barrier and important extensions that incorporate the image interaction were later developed by Murphy, Good, Young and Müller.[3–5] The current density J achieved from a flat metal that is subject to an external field F is given within this model by $J_{\mathrm{FN}} = at^{-2}\phi^{-1}F^2\exp[-bv\phi^{3/2}/F]$, where $a = 1.541434 \times 10^{-6}$ A eV V^{-2} and $b = 6.830890$ eV$^{-3/2}$ V nm^{-1}.[6] ϕ is the work function of the emitter. v and t are tabulated functions that account for the image interaction (they depend on F and ϕ only).[4,7] This equation is referred to as the "standard Fowler-Nordheim equation". To account for the temperature T of the emitter, J_{FN} is actually multiplied by $(\pi k_{\mathrm{B}}T/d)/\sin(\pi k_{\mathrm{B}}T/d)$, where $d = \hbar eF/(2\sqrt{2m\phi}t)$ (k_{B} is Boltzmann's constant, e is the elementary positive charge, m the mass of the electron, $\hbar = h/2\pi$ with h the constant of Planck).[3,4] The temperature-dependent expression is then given by $J_{\mathrm{MG}} = \frac{\pi k_{\mathrm{B}}T/d}{\sin(\pi k_{\mathrm{B}}T/d)} \times at^{-2}\phi^{-1}F^2\exp[-bv\phi^{3/2}/F]$. This is the "Murphy-Good expression".

It is noticeable that the Fermi energy E_{F} that characterizes the free-electron model used for the emitter does not appear in the standard Fowler-Nordheim equation (by Fermi energy, we mean the *energy difference* between the chemical potential μ and the reference level V_{ref} in the emitter; E_{F} corresponds to the kinetic energy of electrons at the chemical potential μ). The Fermi energy E_{F} is related to the density n of free carriers in the emitter by the relation $n = \frac{1}{3\pi^2}(\frac{2mE_{\mathrm{F}}}{\hbar^2})^{3/2}$ and one would expect the emission current to depend on this electron density in an explicit way. The fact E_{F} does not appear in the standard Fowler-Nordheim equation actually results from a series of approximations.[4] The assumption that

the emitter is a *free-electron* metal (kinetic energy given by $\frac{1}{2m}(p_x^2+p_y^2+p_z^2)$, with p_x, p_y and p_z the components of the momentum) enables the current density J to be written in the form $J = e\int_{V_{\rm ref}}^{\infty} N(W)D(W)dW$. In this expression, $W = E - \frac{1}{2m}(p_x^2+p_y^2)$ is the normal-component of the electron energy (E is the total energy) and $N(W) = \frac{4\pi mkT}{h^3}\ln\{1+\exp[-(W-\mu)/k_{\rm B}T]\}$ is the supply function ($N(W)dW$ represents the number of incident electrons per unit surface and per unit of time, with normal energy between W and $W+dW$). $D(W)$ is the transmission coefficient of the surface barrier at the normal-energy W. The supply function $N(W)$ does not depend on the Fermi energy $E_{\rm F}$ (the chemical potential μ is fixed by the vacuum level $V_{\rm vacuum}$ and the work function ϕ by $\mu = V_{\rm vacuum} - \phi$; it is the reference level $V_{\rm ref} = V_{\rm vacuum} - \phi - E_{\rm F}$ that is affected by the particular value of $E_{\rm F}$; $N(W)$ however only depends on $W-\mu$). Any dependence of the current density J on the Fermi energy $E_{\rm F}$ must therefore come from the transmission coefficient $D(W)$. Within the standard Fowler-Nordheim theory, $D(W)$ is however calculated using the simple Jeffreys[8]-Wentzel[9]-Kramers[10]-Brillouin[11] (JWKB) approximation $D(W) = \exp\{-\frac{2\sqrt{2m}}{\hbar}\int_{z_1}^{z_2}[V(z)-W]^{1/2}dz\}$, where z_1 and z_2 refer to the classical turning points of the potential barrier $V(z)$ at the normal-energy W. Within this approximation, $D(W)$ only depends on $V(z)-W$ in the tunneling part of the barrier and is therefore also independent of the Fermi energy $E_{\rm F}$.

It is known from previous work that the simple JWKB approximation does not provide the exact solution for the electronic transmission $D(W)$ and that corrections in the form of an effective prefactor $P_{\rm eff}$ must be considered in order to match the exact quantum-mechanical result.[12,13] Since the standard Fowler-Nordheim (FN) theory relies on this JWKB approximation, it is also necessary to include a correction factor $\lambda^{\rm MG}$ in the (modified) Murphy-Good expression $J_{\rm MG} = \lambda^{\rm MG}\times\frac{\pi k_{\rm B}T/d}{\sin(\pi k_{\rm B}T/d)}\times at^{-2}\phi^{-1}F^2\exp[-bv\phi^{3/2}/F]$ in order to match the exact result.[12,14] Previous work only addressed the dependence of these correction factors on the electric field F and on the work function ϕ. We show in this work that the Fermi energy $E_{\rm F}$ also contributes significantly to these corrections and we therefore propose a correction factor $\lambda^{\rm MG}$ for the Murphy-Good expression that accounts for the electric field F, for the work function ϕ and for the Fermi energy $E_{\rm F}$ of the emitter. This paper is organized according to the following lines. In Sec. II, we present the transfer-matrix (TM) technique that enables the quantum-mechanical calculation of current densities. In Sec. III, we consider the current densities achieved when the transmission probabilities are calculated using the JWKB approximation. This aims at pointing the effects of some

approximations in the standard Fowler-Nordheim theory and at demonstrating that any dependence of the emission current on the Fermi energy $E_{\rm F}$ is necessarily associated with a more exact calculation of the transmission probabilities. In Sec. IV, we finally investigate the influence of the Fermi energy $E_{\rm F}$ on the current densities one obtains when using the transfer-matrix technique for the calculation of the transmission probabilities. We prove that the emission currents actually *depend* on the Fermi energy $E_{\rm F}$ and we establish the form of this dependence. We finally propose a correction factor $\lambda^{\rm MG}$ that enables the (modified) Murphy-Good expression to provide the exact result for the electronic emission achieved from a flat metal.

II. METHODOLOGY

For the quantum-mechanical modeling of field emission, we consider a system with three regions: (i) Region I ($z \leq 0$), which stands for the metal that provides the electrons, (ii) Region II ($0 \leq z \leq D$), which describes the surface barrier of the emitter, and (iii) Region III ($z \geq D$), which stands for the vacuum region in which the electrons are transmitted. We assume that a bias V is established across Region II, so that the external field F is actually defined by $F = V/D$. For a given value of the bias V, we control the strength of this field by adapting the distance D. We then define $V_{\rm I} = eV - \phi - E_{\rm F}$ as the potential energy in Region I, $V(z) = eV - eFz - \frac{1}{16\pi\epsilon_0}\frac{e^2}{z}$ as the potential energy in Region II (ϵ_0 is the electric constant), and $V_{\rm III} = 0$ as the potential energy in Region III. The potential energy $V(z)$ used in Region II is generally referred to as a "Schottky-Nordheim barrier" (it is depicted in Fig. 1). The barrier is actually prevented from going to $-\infty$ as $z \to 0$ by using the potential energy $V_{\rm I}$ in Region I as lower limit. The calculations presented in this work were achieved using a bias V of 150 V. This value is sufficient to have Region II capture the part of the Shottky-Nordheim barrier that has a significant influence on the emission currents.

Since the potential barrier is varying along one dimension only, one can actually treat the scattering problem in cartesian coordinates. In order to work with a finite set of boundary states, we assume that the wave function is periodic along the x and y coordinates (we take a value of L=10 nm for this lateral periodicity). The boundary states in Region I and III are then defined by

$$\Psi_{i,j}^{\rm I/III,\pm}(\mathbf{r},t) \;=\; e^{i(k_{{\rm x},i}x+k_{{\rm y},j}y)}e^{\pm i\sqrt{\frac{2m}{\hbar^2}(E-V_{\rm I/III})-k_{{\rm x},i}^2-k_{{\rm y},j}^2}z}e^{-iEt/\hbar}, \qquad (1)$$

where $k_{x,i} = i\frac{2\pi}{L}$ and $k_{y,j} = j\frac{2\pi}{L}$. E refers to the total electron energy and the $\pm$ signs to the propagation direction relative to the z-axis.

The next step consists in propagating these boundary states across Region II. Since the barrier is independent of x and y, there is no coupling between states associated with different values of i or j and one can consider the propagation of these different states separately. The idea of the method consists in assuming that the potential energy $V(z)$ in Region II varies in steps along the direction z. For each step Δz, the solutions of Schrödinger's equation are simple plane waves (possibly decaying in the tunneling part of the barrier). The propagation of these boundary states across Region II is then achieved by matching continuity conditions for the wave function and its derivative when going from one side of Region II to the other across these different steps.[13] One can get arbitrarily close to the exact potential barrier by letting $\Delta z \to 0$ (we took Δz=0.001 nm).

This propagation step finally leads to a set of scattering solutions of the form

$$\Psi^{+}_{i,j} \overset{z\leq 0}{\equiv} \Psi^{\mathrm{I},+}_{i,j} + S^{-+}_{(i,j),(i,j)}\Psi^{\mathrm{I},-}_{i,j} \overset{z\geq D}{\equiv} S^{++}_{(i,j),(i,j)}\Psi^{\mathrm{III},+}_{i,j}, \tag{2}$$

which correspond to single incident states $\Psi^{\mathrm{I},+}_{i,j}$ in Region I. The coefficients $S^{++}_{(i,j),(i,j)}$ and $S^{-+}_{(i,j),(i,j)}$ provide the amplitudes of the transmitted and reflected states for the incident state $\Psi^{\mathrm{I},+}_{i,j}$ in Region I. The current density provided by the metal in Region I is then obtained by integrating the contributions of these different scattering solutions. Referring to previous work for technical details,[15,19] the result is given by

$$J_{\mathrm{TM}} = \frac{1}{L^2}\frac{2e}{h}\int_{V_{\mathrm{I}}}^{\infty}\sum_{i,j} f(E)\frac{v_{\mathrm{III},(i,j)}}{v_{\mathrm{I},(i,j)}}|S^{++}_{(i,j),(i,j)}|^2 dE, \tag{3}$$

where the summation is restricted to states that are propagative in Regions I and III. In this expression, $v_{\mathrm{I/III},(i,j)} = \frac{\hbar}{m}\sqrt{\frac{2m}{\hbar^2}(E - V_{\mathrm{I/III}}) - k^2_{\mathrm{x},i} - k^2_{\mathrm{y},j}}$ refers to the group velocity of the incident and transmitted states. $f(E) = 1/\{1+\exp[(E-\mu)/k_{\mathrm{B}}T]\}$ is the Fermi factor, with $\mu = eV - \phi$ the chemical potential. The integration in Eq. 3 is achieved using a step ΔE of 0.025 eV. A room temperature T of 300 K was assumed in this work.

We note finally that the fact there is no coupling between states associated with different i or j makes the matrices $\mathbf{S}^{++}$ and $\mathbf{S}^{-+}$ in Eqs 2 and 3 diagonal. One can therefore construct these diagonal elements separately and the whole procedure only involves the manipulation of scalar numbers. In contrast the transfer-matrix technique presented in previous work for the consideration of three-dimensional problems[15–20] involves the manipulation of large

matrices, which requires more significant computational resources. This formulation of the TM technique takes full advantage of this translational invariance. It is more straightforward to implement, requires much less computational resources and could therefore provide more accurate results.

III. EXAMINATION OF THE CURRENT DENSITIES ACHIEVED WITHIN THE JWKB APPROXIMATION

Within the JWKB approximation, the transmission coefficient of the surface barrier is estimated by $\exp\{-\frac{2\sqrt{2m}}{\hbar}\int_{z_1}^{z_2}[V(z)-E+\frac{\hbar^2}{2m}(k_{\mathrm{x},i}^2+k_{\mathrm{y},j}^2)]^{1/2}dz\}$, with z_1 and z_2 the classical turning points at the normal energy $E-\frac{\hbar^2}{2m}(k_{\mathrm{x},i}^2+k_{\mathrm{y},j}^2)$. In order to highlight the effects of a quantum-mechanical evaluation of this transmission coefficient, it is interesting to first consider the results one would obtain from the expression

$$J_{\mathrm{JWKB}}=\frac{1}{L^2}\frac{2e}{h}\int_{V_{\mathrm{I}}}^{\infty}\sum_{i,j}f(E)e^{-\frac{2\sqrt{2m}}{\hbar}\int_{z_1}^{z_2}[V(z)-E+\frac{\hbar^2}{2m}(k_{\mathrm{x},i}^2+k_{\mathrm{y},j}^2)]^{\frac{1}{2}}dz}dE, \tag{4}$$

in which the transmission through the surface barrier is computed from the JWKB approximation (the turning points z_1 and z_2 depend on the specific values of i, j and E). Since the standard Fowler-Nordheim theory is established within the same approximation, the results obtained for J_{JWKB} should actually match those provided by the Murphy-Good expression $J_{\mathrm{MG}}=\frac{\pi k_{\mathrm{B}}T/d}{\sin(\pi k_{\mathrm{B}}T/d)}\times at^{-2}\phi^{-1}F^2\exp[-bv\phi^{3/2}/F]$ if the integration over states achieved within this theory were exact. This is however not the case. The integration achieved within the standard FN theory relies on a series expansion of the JWKB transmission probabilities and the result of this integration is therefore not exact.[3,4] The discrepancy is well illustrated by Fig. 2, where we represented the ratio $\lambda=J_{\mathrm{JWKB}}/J_{\mathrm{MG}}$ between the results provided by a numerical integration of the JWKB transmission probabilities (i.e., Eq. 4) and the results provided by the Murphy-Good expression. The results correspond to fields F that range between 1 V/nm and 10 V/nm and to work functions ϕ that range between 1.5 eV and 5 eV. The representation is restricted to fields F that keep below the critical value $F_{\mathrm{crit}}=\frac{4\pi\epsilon_0\phi^2}{e^3}$ at which the apex of the surface barrier meets the Fermi level of the metal. The Fermi energy E_{F} was given a value of 10 eV, but both J_{JWKB} and J_{MG} turn out to be insensitive to this parameter for realistic values of E_{F} between 5 and 20 eV.

Fig. 2 is actually representative of the error that characterizes the integration of the

JWKB transmission probabilities (independently of the accuracy of this approximation) in the standard Fowler-Nordheim theory. For a typical field F of 5 V/nm and for a typical work function ϕ of 4.5 eV, this error is of the order of 5%. Increasing the field F or decreasing the work function ϕ both tend to increase this error (errors of the order of 10% are actually achieved). The reason comes from the fact increasing F or decreasing ϕ both tend to increase the width of the normal-energy distribution of the emitted electrons. This reduces the reliability of the Taylor-expansion of the transmission coefficient, which is the quantity that is actually integrated within the standard FN theory.[3,4] We note finally that the results presented in Fig. 2 are independent of the particular value of the Fermi energy $E_{\rm F}$. This confirms the fact that, for realistic values of $E_{\rm F}$ between 5 and 20 eV, any dependence of the current density J on the Fermi energy $E_{\rm F}$ must necessarily be associated with a more exact evaluation of the transmission probabilities (the JWKB approximation used so far leads to results that are independent of $E_{\rm F}$).

IV. EXAMINATION OF THE CURRENT DENSITIES ACHIEVED USING THE TRANSFER-MATRIX TECHNIQUE

We can now consider the current densities $J_{\rm TM}$ obtained using the transfer-matrix technique. In contrast with the results achieved within the JWKB approximation, the current densities $J_{\rm TM}$ given by Eq. 3 provide the exact quantum-mechanical solution for the emission achieved from a flat metal (we checked that the parameters L and V are sufficiently large, and the parameters Δz and ΔE sufficiently small, to enable results with at least three significant digits). The coefficient $\frac{v_{{\rm III},(i,j)}}{v_{{\rm I},(i,j)}}|S^{++}_{(i,j),(i,j)}|^2$ in Eq. 3 is the quantum-mechanical equivalent of the "transmission coefficient" for the incident states $\Psi^{{\rm I},+}_{i,j}$ in Region I. It will depend on the Fermi energy $E_{\rm F}$ for different reasons. Through $V_{\rm I} = eV - \phi - E_{\rm F}$, it enters indeed the definition of $v_{{\rm I},(i,j)}$. It also enters the definition of the boundary states $\Psi^{{\rm I},+}_{i,j}$ in Region I, which affects the results obtained for the coefficients $S^{++}_{(i,j),(i,j)}$. The summation in 3 finally extends over the different propagative states in Region I. Their number is also dependent on $E_{\rm F}$. One can therefore rightly expect the current density $J_{\rm TM}$ to depend on the Fermi energy $E_{\rm F}$.

To demonstrate that the emission current $J_{\rm TM}$ indeed depends on the Fermi energy $E_{\rm F}$, we represented in Figs 3 and 4 the ratio $\lambda^{\rm MG} = J_{\rm TM}/J_{\rm MG}$ between the current den-

sities J_{TM} provided by the transfer-matrix technique when considering E_{F}=5 eV (Fig. 3) and E_{F}=15 eV (Fig. 4) and the results of the Murphy-Good expression $J_{\mathrm{MG}} = \frac{\pi k_{\mathrm{B}} T/d}{\sin(\pi k_{\mathrm{B}} T/d)} \times a t^{-2} \phi^{-1} F^2 \exp[-bv\phi^{3/2}/F]$. The results correspond again to fields F that range between 1 V/nm and 10 V/nm and to work functions ϕ that range between 1.5 eV and 5 eV. The quantum-mechanical results J_{TM} turn out to exhibit a significant dependence on the Fermi energy E_{F}. The emission currents achieved for E_{F}=5 eV are *larger* than those achieved for E_{F}=15 eV. We also observe that the quantum-mechanical results J_{TM} deviate more significantly than J_{JWKB} from the Murphy-Good expression J_{MG}. A closer agreement between J_{TM} and J_{MG} is however achieved at low fields F and at high work functions ϕ. This corresponds indeed to conditions where the JWKB approximation provides a better estimation of the tunneling probabilities.[13] The way these tunneling probabilities are integrated within the standard Fowler-Nordheim theory is also more reliable in these conditions. The fact the current densities J_{TM} achieved for E_{F}=5 eV are larger than those achieved for E_{F}=15 eV is contrary to what one would expect since a smaller Fermi energy E_{F} also corresponds to a smaller density n of free carriers in the metal. For realistic values of E_{F} between 5 and 20 eV, the Fermi energy E_{F} actually only influence the current density J_{TM} through the transmission probabilities. These transmission probabilities appear in our results to be higher for smaller values of the Fermi energy E_{F}. For values of E_{F} smaller than the typical width of the normal-energy distribution of the emitted electrons, there is however a cut-off of this distribution and the emission current J_{TM} will finally decrease.

This is illustrated in Fig. 5 where we represented the ratio $\lambda^{\mathrm{MG}} = J_{\mathrm{TM}}/J_{\mathrm{MG}}$ achieved when considering a Fermi energy E_{F} with values between 0 and 20 eV. This result corresponds to a typical field F of 5 V/nm and a typical work function ϕ of 4.5 eV. For $E_{\mathrm{F}} \leq$1.6 eV, there is a cut-off of the normal-energy distribution of the emitted electrons by the reference potential $V_{\mathrm{ref}} = eV - \phi - E_{\mathrm{F}}$ of the emitter. We find as expected that $J_{\mathrm{TM}} \rightarrow 0$ as $E_{\mathrm{F}} \rightarrow 0$, which is the result expected since the density n of free carriers also tends to zero in this limit. We note that the Murphy-Good expression J_{MG} does not account for a possible cut-off of the energy-distribution by the reference potential of the emitter; it is indeed established within the assumption that $V_{\mathrm{ref}} \rightarrow -\infty$. For E_{F} >1.6 eV, the reference potential of the emitter is sufficiently low to let any cut-off of the normal-energy distribution of the emitted electrons have a negligible impact. The emission current depends then on E_{F} through the quantum-mechanical transmission probabilities and we observed that these

transmission probabilities decrease with $E_{\rm F}$.

For practical purposes, one is interested by the correction factor $\lambda^{\rm MG}$ to use with the (modified) Murphy-Good expression $J_{\rm MG} = \lambda^{\rm MG} \times \frac{\pi k_{\rm B} T/d}{\sin(\pi k_{\rm B} T/d)} \times a t^{-2} \phi^{-1} F^2 \exp[-bv\phi^{3/2}/F]$ in order to match the quantum-mechanical result $J_{\rm TM}$. For realistic values of $E_{\rm F}$ between 5 and 20 eV, the data represented in Fig. 5 turns out to be very well represented by the expression $\lambda^{\rm MG} = J_{\rm TM}/J_{\rm MG} = 0.878959 - 0.454183 \times 10^{-1} Z + 0.344853 \times 10^{-2} Z^2 - 0.277680 \times 10^{-3} Z^3 + 0.190602 \times 10^{-4} Z^4 - 0.651969 \times 10^{-6} Z^5$, where $Z = E_{\rm F} - 10$ with $E_{\rm F}$ the Fermi energy in eV. This adjustment is characterized by a mean absolute error of 1.0×10^{-4} and a maximal absolute error of 3.9×10^{-4}. The entire set of $\lambda^{\rm MG}$ data that correspond to fields F between 1 V/nm and 10 V/nm, to work functions ϕ between 1.5 eV and 5 eV, and to Fermi energies $E_{\rm F}$ between 5 and 20 eV can be represented by a polynomial adjustment of the form $\lambda^{\rm MG} = J_{\rm TM}/J_{\rm MG} = \sum_{i=0}^{5} \sum_{j=0}^{5} \sum_{k=0}^{5} a_{ijk} X^i Y^j Z^k$, where $X = F - 5$ with F the field strength in V/nm, $Y = \phi - 3.5$ with ϕ the work function in eV and $Z = E_{\rm F} - 10$ with $E_{\rm F}$ the Fermi energy in eV. The coefficients a_{ijk} are provided in Table I. This expression is restricted to fields F that keep below $F_{\rm crit} = \frac{4\pi\epsilon_0 \phi^2}{e^3}$. It provides a mean absolute error of 4.6×10^{-4} on the exact data, with a maximal absolute error of 1.2×10^{-2}. This expression extends previous work by accounting for the Fermi energy $E_{\rm F}$ in addition to the field F and the work function ϕ already considered.

V. CONCLUSION

We used a transfer-matrix technique to simulate field electron emission from a flat metal. The objective was to confront the results of an exact quantum-mechanical scheme with those provided by the standard Fowler-Nordheim equation. We investigated in particular the dependence of the emission current on the Fermi energy $E_{\rm F}$ of the emitter. This parameter, which is related to the density n of free carriers in the emitter, does not appear in the standard Fowler-Nordheim equation. Within the usual free-electron description of the emitter and as soon as $E_{\rm F}$ exceeds the typical width of the normal-energy distribution of the emitted electrons, it turns out that the Fermi energy $E_{\rm F}$ only influences the emission currents trough the probability that incident electrons have to cross the surface barrier of the emitter. Within the standard Fowler-Nordheim theory, the transmission probabilities are calculated using the simple JWKB approximation, which does not account for the particular

value of E_{F}. This dependence however appears when an exact quantum-mechanical scheme is used for their calculation and we established the form of this dependence. We finally proposed a polynomial adjustment for the correction factor λ^{MG} to use with the Murphy-Good expression in order to match the exact quantum-mechanical result. This correction factor accounts for the field, for the work function and for the Fermi energy of the emitter.[21] It should be useful for the analysis of field-emission data.

Acknowledgments

This work was funded by the National Fund for Scientific Research (FNRS) of Belgium. It used resources of the Interuniversity Scientific Computing Facility located at the University of Namur, Belgium, which is supported by the F.R.S.-FNRS under convention No. 2.4617.07. The author is grateful to R.G. Forbes for useful discussions.

---

[1] N.S. Xu and S.E. Huq, Mat. Sci. Eng. R **48**, 47 (2005).

[2] R.H. Fowler and L.W. Nordheim, Proc. R. Soc. London Ser. A **119**, 173 (1928).

[3] E.L. Murphy and R.H. Good, Phys. Rev. **102**, 1464 (1956).

[4] R.H. Good and E.W. Müller, *Handbuch der Physik* (S. Flugge, Springler Verlag, Berlin) **21**, 176 (1956).

[5] R.D. Young, Phys. Rev. **113**, 110 (1959).

[6] R.G. Forbes, J. Vac. Sci. Technol. B **26**, 788 (2008).

[7] R.G. Forbes and J.H.B. Deane, Proc. R. Soc. A **463**, 2907 (2007).

[8] H. Jeffreys, Proc. London Math. Soc. **23**, 428 (1925).

[9] G. Wentzel, Z. Phys. **38**, 518 (1926).

[10] H.A. Kramers, Z. Phys. **33**, 828 (1926).

[11] L. Brillouin, Compt. Rend. **183**, 24 (1926).

[12] R.G. Forbes, J. Appl. Phys. **103**, 114911 (2008).

[13] A. Mayer, J. Phys.-Condens. Matter **22**, 175007 (2010).

[14] A. Mayer, J. Vac. Sci. Technol. B **28**, 758 (2010).

[15] A. Mayer and J.-P. Vigneron, Phys. Rev. B **56**, 12599 (1997).

[16] A. Mayer and J.-P. Vigneron, Phys. Rev. E **59**, 4659 (1999).

[17] A. Mayer and J.-P. Vigneron, Phys. Rev. E **60**, 7533 (1999).

[18] A. Mayer and J.-P. Vigneron, Phys. Rev. E **61**, 5953 (2000).

[19] A. Mayer, M.S. Chung, B.L. Weiss, N.M. Miskovsky and P.H. Cutler, Phys. Rev. B **78**, 205404 (2008).

[20] A. Mayer A and Ph. Lambin, Nanotechnology **16**, 2685 (2005).

[21] An application that implements this polynomial adjustment can be found on the author's web page at http://perso.fundp.ac.be/∼amayer/FieldEmission

a_{ijk}		$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$k=0$	$i=0$	0.69828	0.20957	-0.47806E-01	0.28919E-01	-0.11910E-01	0.18452E-02
$k=0$	$i=1$	-0.24140E-01	0.33287E-01	-0.28281E-01	0.77466E-02	0.46585E-02	-0.24908E-02
$k=0$	$i=2$	-0.41751E-02	0.68682E-02	0.11466E-03	-0.68572E-02	0.42948E-02	-0.62673E-03
$k=0$	$i=3$	-0.32921E-03	-0.10277E-02	0.24878E-02	-0.18029E-02	0.60355E-03	-0.10757E-03
$k=0$	$i=4$	0.93652E-04	-0.37783E-03	0.33822E-03	-0.89777E-04	0.34451E-04	-0.26123E-04
$k=0$	$i=5$	0.14088E-04	-0.10958E-05	0.14137E-05	-0.18512E-04	0.65233E-05	0.26640E-05
$k=1$	$i=0$	-0.35324E-01	-0.13648E-01	0.48816E-02	-0.15105E-02	0.10822E-03	0.91904E-04
$k=1$	$i=1$	0.12859E-02	-0.23861E-02	0.15198E-02	0.10195E-03	-0.57578E-03	0.18859E-03
$k=1$	$i=2$	0.30497E-03	-0.37680E-03	-0.16651E-03	0.51745E-03	-0.30153E-03	0.54098E-04
$k=1$	$i=3$	0.17613E-04	0.79291E-04	-0.16880E-03	0.12595E-03	-0.50957E-04	0.10882E-04
$k=1$	$i=4$	-0.66970E-05	0.24559E-04	-0.24491E-04	0.12125E-04	-0.40207E-05	0.86719E-06
$k=1$	$i=5$	-0.10734E-05	0.92980E-06	-0.10770E-05	0.11006E-05	-0.18004E-06	-0.13294E-06
$k=2$	$i=0$	0.29717E-02	0.88654E-03	-0.56506E-03	0.17008E-03	-0.37878E-05	-0.12927E-04
$k=2$	$i=1$	-0.82806E-04	0.23950E-03	-0.17379E-03	-0.79996E-06	0.64499E-04	-0.23527E-04
$k=2$	$i=2$	-0.30783E-04	0.41819E-04	0.14178E-04	-0.53714E-04	0.32615E-04	-0.61460E-05
$k=2$	$i=3$	-0.25595E-05	-0.69096E-05	0.18877E-04	-0.15033E-04	0.46453E-05	-0.36513E-06
$k=2$	$i=4$	0.67108E-06	-0.26075E-05	0.26672E-05	-0.11568E-05	0.24389E-06	-0.35676E-07
$k=2$	$i=5$	0.13198E-06	-0.13956E-06	0.50418E-07	-0.21722E-07	0.22443E-07	-0.75094E-08
$k=3$	$i=0$	-0.29146E-03	-0.36451E-04	0.74498E-04	-0.28661E-04	0.26987E-05	0.13343E-05
$k=3$	$i=1$	0.37305E-05	-0.28012E-04	0.25052E-04	-0.21854E-05	-0.71270E-05	0.26613E-05
$k=3$	$i=2$	0.37434E-05	-0.60553E-05	-0.76292E-06	0.64302E-05	-0.39925E-05	0.71242E-06
$k=3$	$i=3$	0.38643E-06	0.74188E-06	-0.22465E-05	0.17774E-05	-0.59560E-06	0.81797E-07
$k=3$	$i=4$	-0.79049E-07	0.32141E-06	-0.33509E-06	0.13714E-06	-0.33716E-07	0.91286E-08
$k=3$	$i=5$	-0.14946E-07	0.15560E-07	-0.72602E-08	0.78641E-08	-0.29691E-08	-0.55529E-09
$k=4$	$i=0$	0.24408E-04	-0.70958E-06	-0.72940E-05	0.33457E-05	-0.55985E-06	-0.51351E-07
$k=4$	$i=1$	-0.21790E-07	0.26266E-05	-0.25867E-05	0.30501E-06	0.51993E-06	-0.16528E-06
$k=4$	$i=2$	-0.36359E-06	0.65033E-06	0.16972E-09	-0.59702E-06	0.36481E-06	-0.57116E-07
$k=4$	$i=3$	-0.35931E-07	-0.75454E-07	0.18892E-06	-0.13620E-06	0.68770E-07	-0.22447E-07
$k=4$	$i=4$	0.74434E-08	-0.30006E-07	0.31091E-07	-0.13269E-07	0.60377E-08	-0.23760E-08
$k=4$	$i=5$	0.94592E-09	-0.58292E-09	0.14936E-08	-0.26918E-08	0.31856E-09	0.52339E-09
$k=5$	$i=0$	-0.94061E-06	0.10566E-06	0.29656E-06	-0.14783E-06	0.32575E-07	-0.13093E-08
$k=5$	$i=1$	-0.45184E-08	-0.10659E-06	0.10731E-06	-0.13137E-07	-0.15472E-07	0.33644E-08
$k=5$	$i=2$	0.15016E-07	-0.27969E-07	0.16739E-08	0.23703E-07	-0.14348E-07	0.20542E-08
$k=5$	$i=3$	0.13327E-08	0.33496E-08	-0.67836E-08	0.44185E-08	-0.33023E-08	0.14496E-08
$k=5$	$i=4$	-0.30239E-09	0.11948E-08	-0.12395E-08	0.57572E-09	-0.35600E-09	0.14780E-09
$k=5$	$i=5$	-0.21803E-10	-0.69976E-11	-0.92565E-10	0.17808E-09	-0.13727E-10	-0.37574E-10

TABLE I: Coefficients a_{ijk} of the polynomial adjustment $\lambda^{\mathrm{MG}} = J_{\mathrm{TM}}/J_{\mathrm{MG}} = \sum_{i=0}^{5}\sum_{j=0}^{5}\sum_{k=0}^{5} a_{ijk}X^iY^jZ^k$ for the prefactor λ^{MG} to use with the Murphy-Good expression in order to match the current densities J_{TM} provided by the transfer-matrix technique. In this expression, $X = F - 5$ with F the field strength in V/nm, $Y = \phi - 3.5$ with ϕ the work function in eV and $Z = E_{\mathrm{F}} - 10$ with E_{F} the Fermi energy in eV. This expression is restricted to 1 V/nm $\leq F \leq$ 10 V/nm, 1.5 eV $\leq \phi \leq$ 5 eV, 5 eV $\leq E_{\mathrm{F}} \leq$ 20 eV and $F < \frac{4\pi\epsilon_0\phi^2}{e^3}$.

FIGURE CAPTIONS

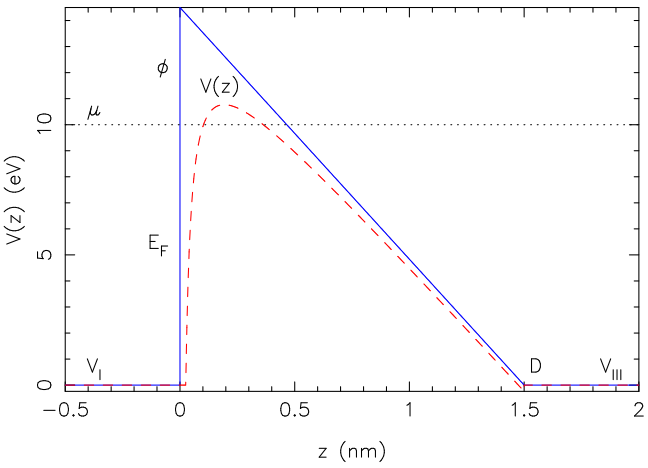
FIG. 1. (Color online) Potential energy for the case of a Schottky-Nordheim barrier $V(z) = eV - eFz - \frac{1}{16\pi\epsilon_0}\frac{e^2}{z}$ (dashed). The solid line represents the potential energy $V(z) = eV - eFz$ that is relevant to the original Fowler-Nordheim theory. The representation corresponds to a Fermi energy $E_{\rm F}$ of 10 eV, a work function ϕ of 4.5 eV and a bias V of 14.5 V (a bias V of 150 V is used for the transfer-matrix calculations).

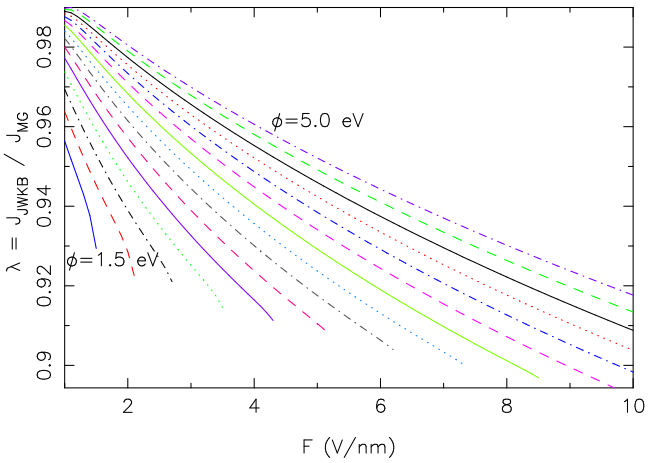
FIG. 2. (Color online) Ratio $\lambda = J_{\rm JWKB}/J_{\rm MG}$ between the current density $J_{\rm JWKB}$ obtained at T=300 K from a numerical integration of the JWKB transmission probabilities (Eq. 4 in the text) and the current density $J_{\rm MG}$ provided by the Murphy-Good expression. The results correspond to fields F that range between 1 V/nm and 10 V/nm. The work function ϕ ranges between 1.5 eV and 5 eV (upwards, by increments of 0.25 eV). These results correspond to a Fermi energy $E_{\rm F}$ of 10 eV, but both $J_{\rm JWKB}$ and $J_{\rm MG}$ turn out to be insensitive to this parameter (for tested values between 5 and 20 eV).

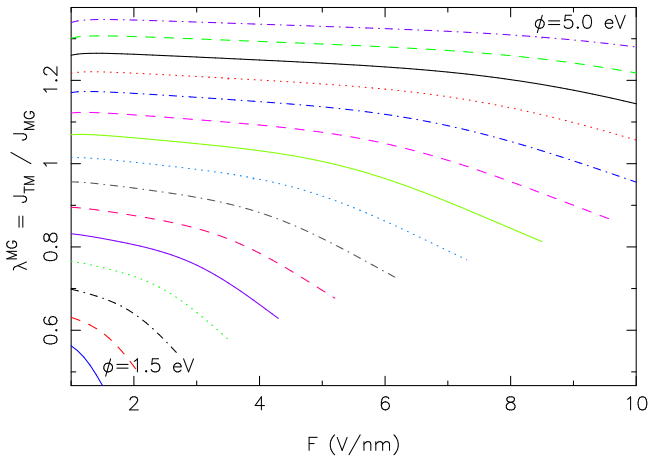
FIG. 3. (Color online) Ratio $\lambda^{\rm MG} = J_{\rm TM}/J_{\rm MG}$ between the current density $J_{\rm TM}$ obtained at T=300 K from a transfer-matrix calculation (Eq. 3 in the text) and the current density $J_{\rm MG}$ provided by the Murphy-Good expression. The results correspond to fields F that range between 1 V/nm and 10 V/nm. The work function ϕ ranges between 1.5 eV and 5 eV (upwards, by increments of 0.25 eV). The results correspond to a Fermi energy $E_{\rm F}$ of 5 eV.

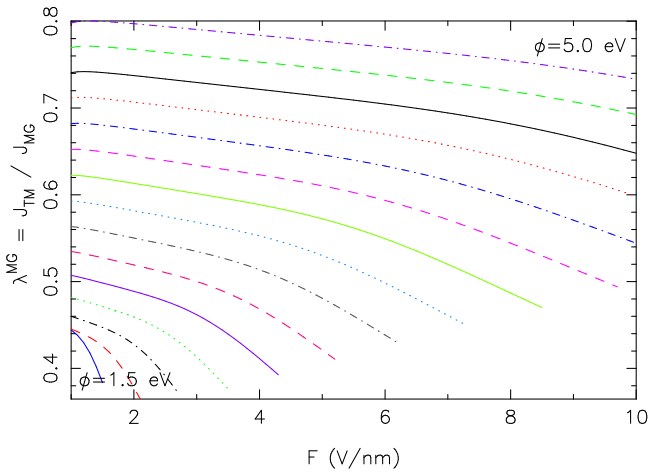
FIG. 4. (Color online) Ratio $\lambda^{\rm MG} = J_{\rm TM}/J_{\rm MG}$ between the current density $J_{\rm TM}$ obtained at T=300 K from a transfer-matrix calculation (Eq. 3 in the text) and the current density $J_{\rm MG}$ provided by the Murphy-Good expression. The results correspond to fields F that range between 1 V/nm and 10 V/nm. The work function ϕ ranges between 1.5 eV and 5 eV (upwards, by increments of 0.25 eV). The results correspond to a Fermi energy $E_{\rm F}$ of 15 eV.

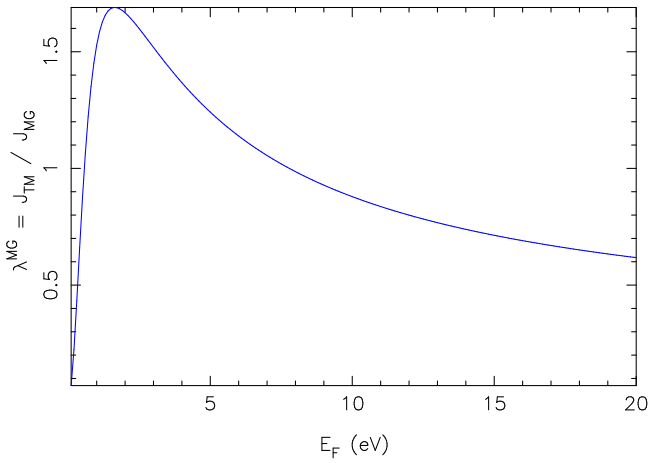
FIG. 5. (Color online) Ratio $\lambda^{\mathrm{MG}} = J_{\mathrm{TM}}/J_{\mathrm{MG}}$ between the current density J_{TM} obtained at T=300 K from a transfer-matrix calculation (Eq. 3 in the text) and the current density J_{MG} provided by the Murphy-Good expression. The results correspond to a field F of 5 V/nm and to a work function ϕ of 4.5 eV. They are presented as a function of the Fermi energy E_{F} of the emitter.


ϕ
V(z)
μ
E_F
V_I
D
V_III
V(z) (eV)
z (nm)
0
5
10
−0.5
0
0.5
1
1.5
2


λ = J_JWKB / J_MG
0.9
0.92
0.94
0.96
0.98
2
4
6
8
10
F (V/nm)
ϕ=5.0 eV
ϕ=1.5 eV


ϕ=5.0 eV
ϕ=1.5 eV
λ^MG = J_TM / J_MG
0.6
0.8
1
1.2
2
4
6
8
10
F (V/nm)


0.8
0.7
0.6
0.5
0.4
$\lambda^{MG} = J_{TM} / J_{MG}$
ϕ=5.0 eV
ϕ=1.5 eV
2
4
6
8
10
F (V/nm)


$\lambda^{MG} = J_{TM} / J_{MG}$
0.5
1
1.5
5
10
15
20
E_F (eV)